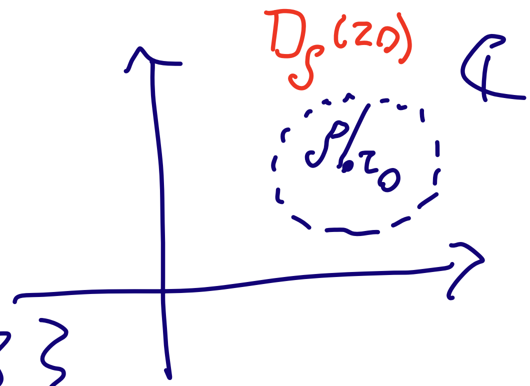
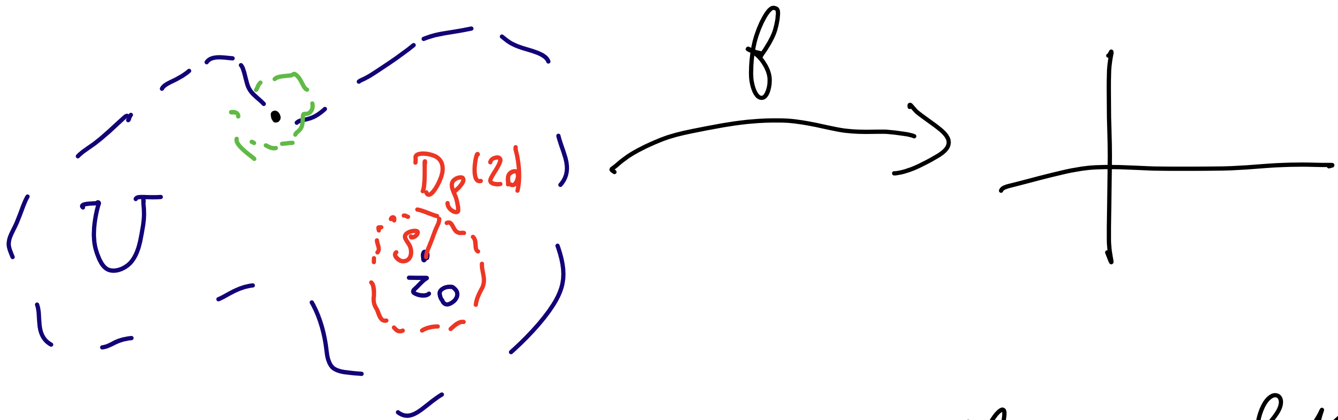
Lecture 3: Ch II Analytic Functions

Limits:

Notation:

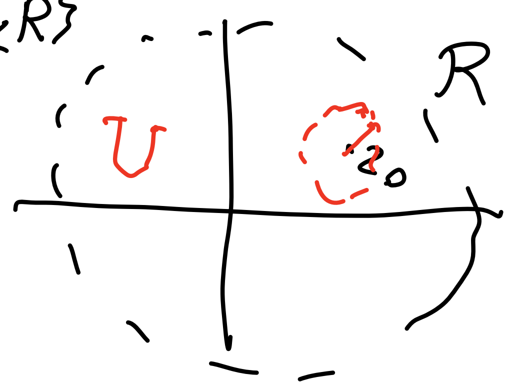
$$D_\rho(z_0) = \{ z \in \mathbb{C} : |z - z_0| < \rho \}$$





Def 1: A subset U of the complex plane $\mathbb{C}$ is OPEN, if for every point z_0 in U, there exists $\rho > 0$, such that the open disk $D_\rho(z_0)$ is contained in U.

Ex: $U = D_R(0) = \{ z : |z| < R \}$ is an open set.

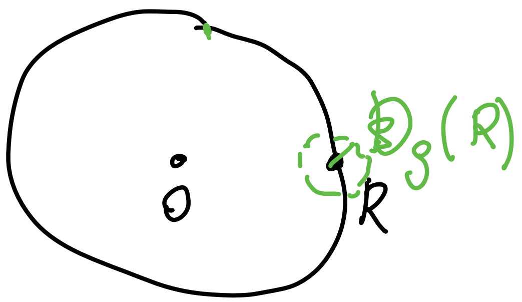


Given z_0 in U, $|z_0| < R$,

Take $\rho := \frac{1}{2}(R - |z_0|)$. Then $D_\rho(z_0) \subset U$

Ex: The closed disk

$T = \{z : |z| \leq R\}$ is not open.



$B_\delta(R)$ is not contained in T as soon as $\delta > 0$.

Let U be an open subset of $\mathbb{C}$.

A complex valued function f, defined over U, is a map

$$f : U \to \mathbb{C}$$

In cartesian coordinates

$$\boxed{f(x+iy) = u(x,y) + i\,v(x,y)},$$

where u, v are real valued functions defined over U.

<u>Ex:</u> Let $U = \mathbb{C}$.

Let $f(z) = z^2$.

$$f(x+iy) = (x+iy)^2 = (x+iy)(x+iy) =$$

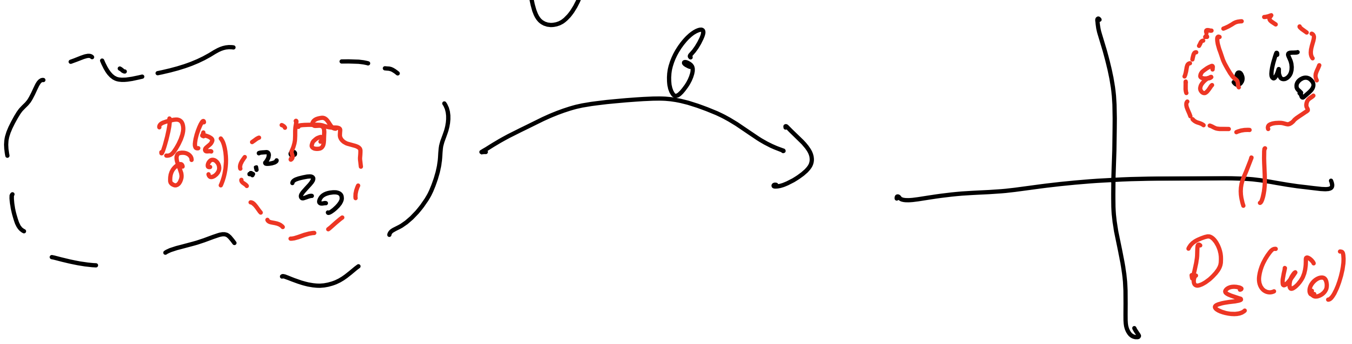
$$= \underbrace{(x^2-y^2)}_{u(x,y)} + i\,\underbrace{(2xy)}_{v(x,y)}$$

<u>Def 2:</u> Let $f: U \to \mathbb{C}$ be a complex valued function defined on an open subset U of $\mathbb{C}$. We say that the limit of $f(z)$, as z approaches z_0, is w_0, and write

$$\boxed{\lim_{z \to z_0} f(z) = w_0,}$$

if

a) (intuitive) $f(z)$ can be made arbitrarily close to w_0, if we choose z sufficiently close to z_0, but $z \neq z_0$.



b) (precise) For every $\varepsilon > 0$, there exists $\delta > 0$, such that if

$0 < |z - z_0| < \delta$, then $|f(z) - w_0| < \varepsilon$.

<u>Def 3</u>: Let f be as above.
Then f is said to be <u>continuous</u> at z_0, if $\lim_{z \to z_0} f(z) = f(z_0)$.

<u>Remark</u>: In cartesian coordinate

If $f(x+iy) = u(x,y) + iv(x,y)$,

then, $\lim_{\substack{(x,y) \to (x_0, y_0) \\ x+iy \qquad x_0+iy_0}} f(z) = w_0 = u_0 + iv_0$

$\Longleftrightarrow$

$$\left(\lim_{(x,y) \to (x_0, y_0)} u(x,y) = u_0 \quad \text{AND} \quad \lim_{(x,y) \to (x_0, y_0)} v(x,y) = v_0 \right).$$

Ex: $f(z) = z^2$.

$$\lim_{z \to 2+3i} f(z) = (2+3i)^2 = (2^2-3^2) + i(12)$$

-5

(i.e., f is continuous at $2+3i$)

Reason: $f(x+iy) = \underbrace{u(x,y)}_{x^2-y^2} + i\underbrace{v(x,y)}_{2xy}$

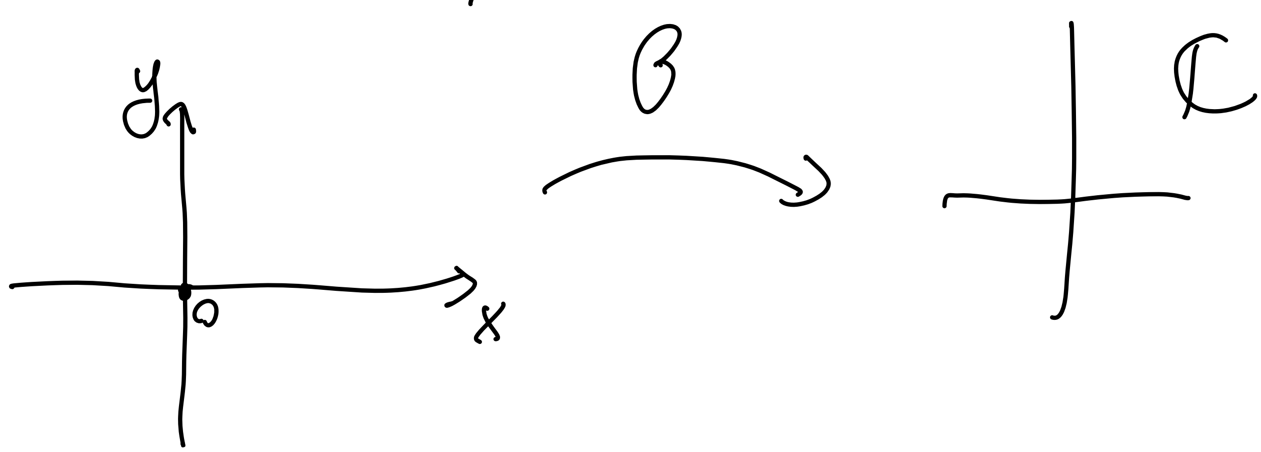
and we we know that u, v are continuous on $\mathbb{C}$, and

so $\lim_{(x,y)\to(2,3)} u(x,y) = 2^2 - 3^2 = -5$

$\lim_{(x,y)\to(2,3)} v(x,y) = 2\cdot 2\cdot 3 = 12$.

Ex: Let $U = \mathbb{C}$; $f(z) = \frac{\bar{z}}{z}$ defined on U except at 0,

$\lim_{z\to 0} f(z)$ does not exist,



Let $z = x + 0i$ approach 0 along the x-axis

$$\lim_{x \to 0} \frac{\bar{x}}{x} = \lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0} 1 = \boxed{1}$$

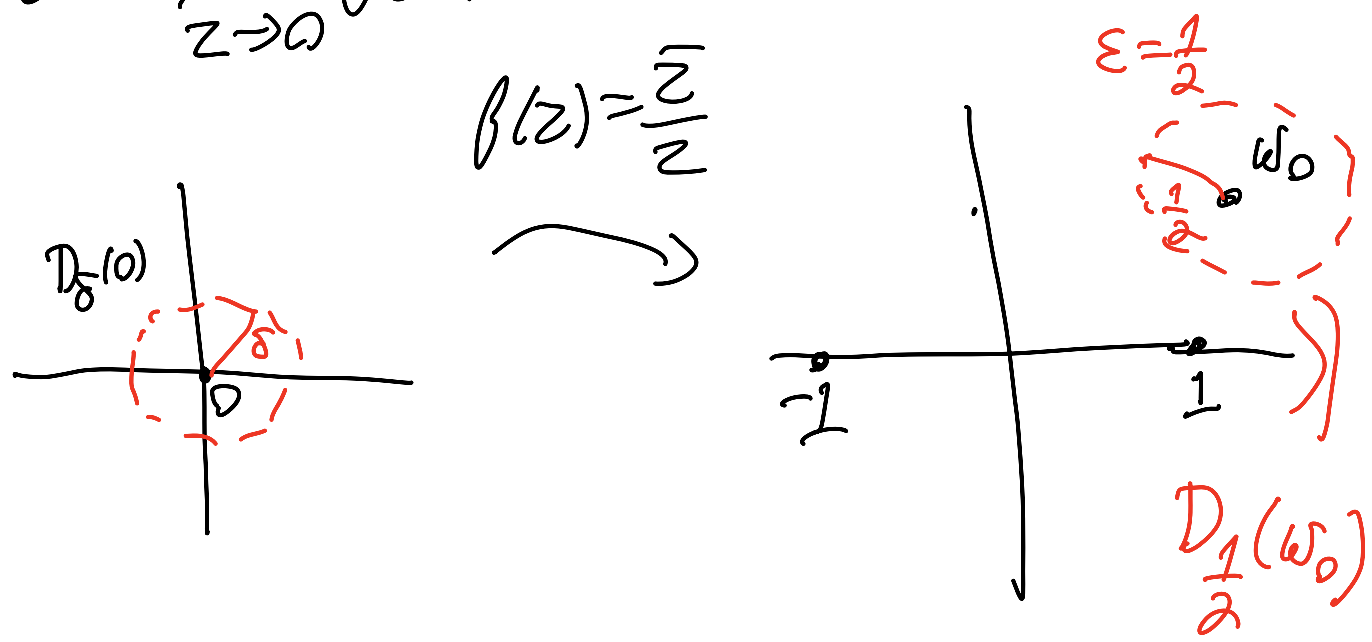
x real

Let z approach 0 along the y-axis. $z = 0 + yi$

$$\lim_{y \to 0} f(iy) = \lim_{y \to 0} \frac{\overline{(iy)}}{iy} =$$

$$= \lim_{y \to 0} \frac{-iy}{iy} = \lim_{y \to 0} -1 = \boxed{-1}.$$

So $\lim_{z \to 0} f(z)$ does not exist.



If $\varepsilon = \frac{1}{2}$, then for every $w_0 \in \mathbb{C}$, $D_{\frac{1}{2}}(w_0)$ does not contain at least one of 1 or -1. For every $\delta > 0$ $D_\delta(0)$ intersect non-zero points of both the x-axis and the y-axis. The values of f at non-zero real cpx numbers is 1 and at non-zero pt on the y-axis, is -1. So both 1 and -1 are values of f at points of $D_\delta(0)$. So

$$f\left(D_\delta(0) \text{ minus } 0\right) \not\subset D_{\frac{1}{2}}(w_0)$$

not contained

The usual limit laws hold, (sum, product law, ...)

Ex

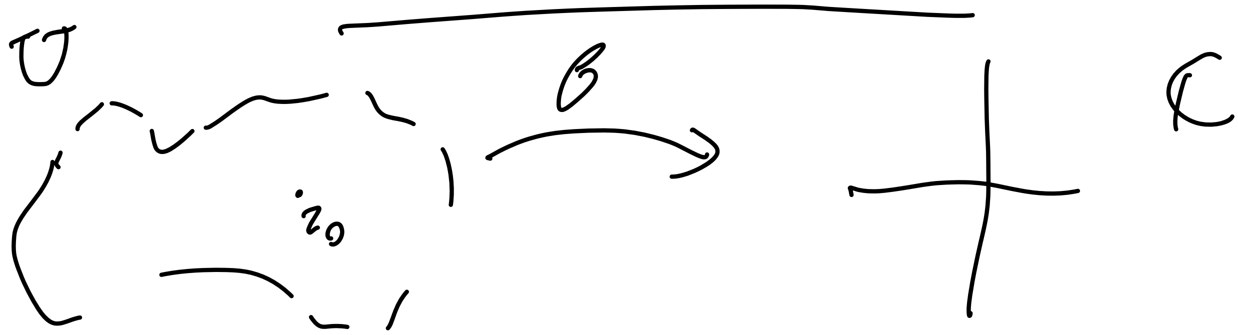
<u>Theorem</u>; (Quotient Limit Thm)

If f, g are complex valued functions defined on an open set containing z_0 and $\lim_{z \to z_0} f(z)$ and $\lim_{z \to z_0} g(z)$ both exists, and the latter is non-zero, then

$$\lim_{z \to 0} \frac{f(z)}{g(z)} = \frac{\lim_{z \to 0} f(z)}{\lim_{z \to 0} g(z)}$$

(in particular, we are stating that the limit exists).

Derivatives



<u>Def 4:</u> Let f be a complex valued function, defined on some open set U in the plane. The derivative $f'(z_0)$ of f at a point z_0 of U is

$$f'(z_0) \overset{\text{def}}{:=} \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

provided the limit exists.

In that case, f is said to be DIFFERENTIABLE at z_0.

<u>Remark:</u> If f is differentiable at z_0 then it is continuous at z_0, because

$$\lim_{z \to z_0} f(z) - f(z_0) = \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) \cdot (z - z_0) =$$

product rule

$$= \underbrace{\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}_{f'(z)} \cdot \underbrace{\lim_{z \to z_0} z - z_0}_{0} = 0.$$

So $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \left(f(z) - f(z_0) \right) + f(z_0) =$

$$0 + f(z_0) = f(z_0). \quad \square$$

<u>Ex</u>: Let $f(z) = \bar{z}$.

$$f(x + iy) = x - iy$$

Then f is not differentiable anywhere. Let us check at $z_0 = 0$. $f(0) = \bar{0} = 0$.

$$\lim_{z \to 0} \frac{f(z) - 0}{z - 0} = \lim_{z \to 0} \frac{\bar{z}}{z} \text{ does not exist.}$$

<u>Ex</u>: $f(z) = z^n$ ($n \in \mathbb{Z}$) is differentiable everywhere, and

$$f'(z_0) = n z_0^{n-1}$$

(except when $n < 0$ and $z_0 = 0$)

This is a consequence of the product Rule.

The usual derivative laws (Theorems) hold.

__Example__: Let f, g be defined on an open set U and differentiable at $z_0 \in U$. Then $f \cdot g$ is differentiable at z_0 and

$$\boxed{(f \cdot g)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0).}$$

__Proof__:

$$\lim_{z \to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0} =$$

$$\lim_{z \to z_0} \frac{[f(z)g(z) - f(z)g(z_0)] + [f(z)g(z_0) - f(z_0)g(z_0)]}{z - z_0}$$

$$= \lim_{z \to z_0} f(z)\frac{g(z) - g(z_0)}{z - z_0} + \lim_{z \to z_0} g(z_0)\frac{f(z) - f(z_0)}{z - z_0}$$

Sum Limit Law

$$= \underbrace{\lim_{z\to z_0} f(z)}_{\substack{\| \\ f(z_0)}} \cdot \underbrace{\lim_{z\to z_0} \left(\frac{g(z)-g(z_0)}{z-z_0}\right)}_{\substack{\| \\ g'(z_0)}} + \underbrace{\lim_{z\to z_0} g(z_0)}_{g(z_0)} \cdot \underbrace{\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}}_{\substack{\| \text{ exists} \\ f'(z_0) \\ \text{by assumption}}}$$

Product Limit Law

f is contin at z_0

$$= f(z_0)\, g'(z_0) + g(z_0)\, f'(z_0). \quad \square$$

Ex: $n > 1$

$$(z^n)' = (z\, z^{n-1}) = \underbrace{z'}_{\substack{\| \\ 1}} \cdot z^{n-1} + \underbrace{z\,\underbrace{(z^{n-1})}_{\substack{\| \text{ by inducti} \\ (n-1)z^{n-2}}}}_{(n-1)z^{n-2}}$$

$$= n z^{n-1} \quad \square$$

The Cauchy-Riemann Equation

Let $f(\underbrace{x+iy}_{z}) = u(x,y) + i v(x,y)$.

We will see that the

assumption that $f'(\underset{\substack{\shortparallel \\ x_0+iy_0}}{z_0})$ exists

implies two equations among the partials

$u_x = v_y$ at (x_0, y_0) and

$u_y = -v_x$ " " .